A Theory of Design Reliability Using Probability and Fuzzy Sets

All process designs are subject to uncertainties which make it impossible to state with complete certainty that a design will work. This study presents a measure of design confidence which considers the nature of the uncertainty and the operability of the process as a whole. This measure, called design reliability, quantifies the likelihood that a design will work. Detailed analyses are presented for three special cases: designs subject to random uncertainties, designs subject to fuzzy uncertainties, and designs subject to both random and fuzzy uncertainties. Practical procedures for estimating design reliability in these special cases are also presented.

William L. Kubic, Jr., Fred P. Stein
Department of Chemical Engineering
Lehigh University
Bethlehem, PA 18015

Introduction

The objective of chemical process design is to specify a flowsheet and process equipment that will perform chemical changes reliably and economically. Because engineering analysis is subject to uncertainty, it is important to ask whether the design will work in spite of all the uncertainties.

The problem of design uncertainty has been addressed in the literature. Most studies consider sensitivity analysis and propagation of errors (Chen et al., 1970; Lashmet and Szczepanski, 1974; Sullivan and Uhl, 1982; Macchietto et al., 1986). Few authors, however, consider the process design as an integrated unit. Freeman and Gaddy (1975) used a probabilistic measure of design confidence, but they considered only a specific example. The flexibility index of Swaney and Grossmann (1985) is a general method of uncertainty analysis that considers the entire process, but it uses a simplified description of the uncertainties.

A more complete description of process uncertainties can be obtained using probability theory and/or fuzzy set theory. Probability is used to describe uncertainties that arise from random behavior. Many uncertainties, such as imprecise data, are random, but others are not. Kubic and Stein (1986) discussed modeling errors as nonrandom uncertainties. Modeling errors are the result of assumptions made in an analysis. Deviations of model predictions from reality are uncertain, but because assumptions are not random, modeling errors are not stochastic. Although probabilty is not applicable to nonrandom modeling errors, fuzzy set theory can be used to describe such errors (Kubic and Stein, 1986). In addition to modeling errors, fuzzy set theory can be used to describe other vaguely defined uncertainties of which feedstock availability is one example.

The problem of process design uncertainty with a more complete description of the uncertainties is considered in this study. The objective is to develop measures of design confidence that reflect the nature of the uncertainty. The theory of fuzzy measures is used to quantify the confidence of a process design that is subject to stochastic and/or fuzzy uncertainties.

Process Design Reliability

A chemical plant is a set of equipment used to process raw materials, chemically and physically, to useful products. The objective of design is to specify equipment and recommend operating conditions for producing the desired changes. The process equipment cannot be changed without capital expenditures, but operating conditions can be changed within the limitations of the equipment. Failure to specify the correct operating conditions does not constitute a failure of the design, but if a set of operating conditions that produces the desired products does not exist, the specified equipment is inadequate and the design is a failure.

To analyze the failure of a design, mathematical models of the process specifications and equipment capacities are required. These models are expressed mathematically as inequality constraints.

$$0 \geq g_i(\boldsymbol{d}, \boldsymbol{u}, \boldsymbol{p}) \tag{1}$$

Correspondence concerning this paper should be addressed to Fred P. Stein. The present address of William L. Kubic, Jr., is Los Alamos National Laboratory, NM 87545.

where

$g_i(\cdot)$ = Constraint i

d = Vector of equipment design parameters

u = Vector of operating conditions

p = Vector of modeling parameters

For a given vector of parameters, there is a set of operating conditions that violates each constraint.

$$F_i = \{u \mid g_i(d, u, p) > 0\} \tag{2}$$

The union of F_i is the set of operating conditions for which the process fails.

$$F_T = \bigcup F_i = \{u \mid \max_i g_i(d, u, p) > 0\} \tag{3}$$

The max $g_i > 0$ implies that at least one constraint is violated for a given u; hence, the vector of operating conditions is infeasible. The design is a failure if the complement of F_T is the empty set.

To prove that the complement of F_T is not the empty set, consider the maximum lower bound on max g_i with respect to the operating conditions.

$$\alpha(d, p) = \inf_w \max_i g_i(d, u, p) \tag{4}$$

If α is greater than zero, the max g_i must be positive for all u; hence, no feasible region exists. If the converse is true—i.e., $\alpha \leq 0$—at least one set of operating conditions exists that satisfies all of the constraints, proving that the complement of F_T is not empty.

The constraints given by the inequality of Eq. 1 define a feasible region of operation for a fixed parameter vector. This region is illustrated for an arbitrary process in Figure 1a. These constraints are not only functions of the operating conditions but also functions of the parameters. Because the parameters contain uncertainty, the size, shape, and even the existence of the feasible region is uncertain. The possible change in the size and shape of the feasible region for a different set of parameters is illustrated in Figure 1b. The nominal operating conditions are not included in the feasible region for this realization of the parameters, illustrating the importance of considering operating flexibility.

Let set P be the set of all possible realizations of the parameter vector. P may be partitioned into two subsets: one consisting of all parameter vectors for which a feasible region exists, and the other consisting of all parameter vectors for which a feasible region does not exist. The subsets are represented as follows.

$$M = \{p \in P \mid \alpha(d, p) \leq 0\} \quad \text{(feasible operating region)} \tag{5}$$

$$N = \{p \in P \mid \alpha(d, p) > 0\} \quad \text{(no feasible operating region)} \tag{6}$$

The object of uncertainty analysis is to measure the size of M and N in a way that reflects the nature and magnitude of the uncertainties.

The certainty to which a given event belongs to a particular set can be quantified by a fuzzy measure (Dubois and Prade, 1980). A fuzzy measure, h, is a function from $\mathcal{P}[X]$, where X is the universe, to [0, 1] such that

1. $h(\emptyset) = 0; h(X) = 1$
2. $\forall A, B \in \mathcal{P}(X)$, if $A \subseteq B$, $h(A) \leq h(B)$
3. If $\forall i \in N, A_i \in \mathcal{P}(X)$ and A_i is monotonic, then $\lim_{i\to\infty} h(A_i) = h(\lim_{i\to\infty} A_i)$

Fuzzy measures will be used in this study to quantify design confidence.

A fuzzy measure of M assigns a numerical value to the certainty that a feasible region exists and the design will work. This quantity will be called design reliability in this study.

$$R = h\{M\} \tag{7}$$

Design unreliability is defined as a fuzzy measure quantifying the certainty that a design will fail.

$$U = h\{N\} \tag{8}$$

The closer design reliability is to unity, the greater the likelihood of success. If design unreliability is near unity, the possibility of failure is great. It should be noted that fuzzy measures of success and failure do not necessarily sum to unity.

Having defined design reliability, it is useful to compare it to the flexibility index of Swaney and Grossmann (1985). The flexibility index is defined as follows.

$$F = \max \delta$$

$$\text{s.t.} \quad \max_{p \in T(\delta)} \min_u \max_i g_i(d, u, p) \leq 0$$

$$T(\delta) = \{p \mid (p^N - \delta\Delta p^-) \leq p \leq (p^N + \delta\Delta p^+)\} \tag{9}$$

Δp^+ and Δp^- are the upper and lower bounds on deviations from the nominal parameter value, p^N. $T(\delta)$ is the largest scaled hyperrectangle that can be inscribed in the feasible region.

The flexibility index considers the region $T(\delta)$, which is a subset of M for feasible values of δ. The maximum size of $T(\delta)$ depends on the active constraints in the solution to Eq. 9. Design changes can be made to the inactive constraints without affecting the flexibility index. Hence, the flexibility index is not unique.

Design reliability, on the other hand, considers the entire feasible region. Because $T(\delta)$ is a subset of M, $h[T(\delta)]$ is a lower bound on design reliability. Likewise, $h[\overline{T}(\delta)]$ is an upper bound on design unreliability. When combined with fuzzy measure theory, the method of Swaney and Grossmann leads to a pessimistic estimate of design reliability.

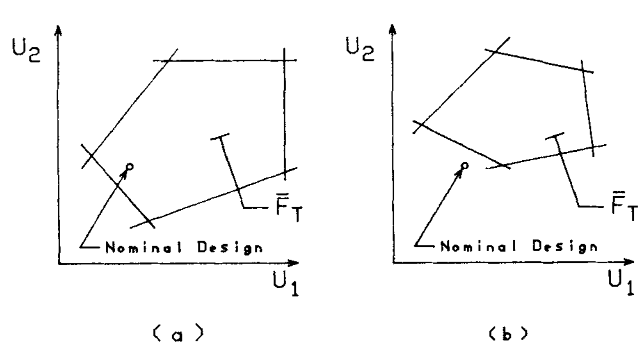

Figure 1. Feasible operating region.
(a) With parameter set 1; (b) With parameter set 2

Design Reliability with Stochastic Uncertainties

To evaluate design reliability, the appropriate fuzzy measure must be selected and evaluated. If all of the uncertainties are stochastic, a probability measure should be used. Probability is a fuzzy measure which, for every A and B such that $A \cap B = \emptyset$, $Pr(A \cup B) = Pr(A) + Pr(B)$.

Design reliability in the special case of random uncertainties is simply the probability that a feasible region exists.

$$R = Pr\{M\} \tag{10}$$

Design unreliability in this case is the probability that a design will fail.

$$U = Pr\{N\} \tag{11}$$

Because M and N are complementary, probabilistic reliability and unreliability sum to unity. Hence, the calculation of unreliability is redundant. Reliability can be expressed in terms of the function $\alpha(\boldsymbol{d}, \boldsymbol{p})$.

$$R = Pr\{\inf_{u} \max_{i} g_i(\boldsymbol{d}, \boldsymbol{u}, \boldsymbol{p}) \leq 0\} \tag{12}$$

It should be noted that design reliability is a function of the vector of equipment design variables, as defined by Eq. 4.

Estimating reliability for processes without operating degrees of freedom

To calculate reliability, the probability distribution for $\alpha(\boldsymbol{d}, \boldsymbol{p})$ must be determined from the distributions of the parameters. In general, $\alpha(\boldsymbol{d}, \boldsymbol{p})$ is a nonlinear, discontinuous function of the parameters. The distribution for $\alpha(\boldsymbol{d}, \boldsymbol{p})$ can be estimated using Monte Carlo methods, but Monte Carlo can be expensive when applied to a process design. This section considers a method of estimating design reliability without using Monte Carlo for the special case of processes with no operating degrees of freedom.

For a process without operating degrees of freedom, Eq. 4 simplifies to the following:

$$\alpha(\boldsymbol{d}, \boldsymbol{p}) = \max_{i} g_i(\boldsymbol{d}, \boldsymbol{p}) \tag{13}$$

The set of parameters that results in failure is written as

$$N = \{\boldsymbol{p} | \max_{i} g_i(\boldsymbol{d}, \boldsymbol{p}) > 0\} \tag{14}$$

The maximum operator in this equation suggests that N can be constructed from several sets. Let N_i be the set of parameter vectors that violates constraint i.

$$N_i = \{\boldsymbol{p} | g_i(\boldsymbol{d}, \boldsymbol{p}) > 0\} \tag{15}$$

N is the union of N_i. Hence, unreliability can be written as follows:

$$U = Pr\{\cup N_i\} \tag{16}$$

Upper and lower bounds on unreliability can be obtained from the probabilities of failure for each individual constraint.

$$\sum_{i=1}^{n} Pr\{N_i\} \geq U \geq \max_{i} Pr\{N_i\} \tag{17}$$

These bounds follow directly from the axioms of probability. The probability of N_i is obtained from the cumulative distribution function of $g_i(\boldsymbol{d}, \boldsymbol{p})$. A linear approximation can be used to obtain a simple estimate of the distribution. However, neglecting nonlinearities can lead to significant errors, as demonstrated by Park and Himmelblau (1980). The moments of a nonlinear function of random variables can be estimated using a quadratic approximation of the function. The distribution function can be determined from the moments. Details of this method are given by Cox (1979) and by Kubic (1986).

The upper bound given by the inequality of Eq. 17 corresponds to disjoint failure modes, and the lower bound is the limiting case of a dominant failure mode that encompasses all other modes of failure. These simple bounds can be useful, but they can also span an unacceptably large range. To develop better bounds on reliability and unreliability it is necessary to consider the fact that the N_i partially overlap.

Consider the function $\mathcal{F}(\boldsymbol{d}, \boldsymbol{p})$, which is defined as follows.

$$\mathcal{F}(\boldsymbol{d}, \boldsymbol{p}) = \prod_{j=1}^{n} - g_j(\boldsymbol{d}, \boldsymbol{p}) \tag{18}$$

This function is negative if an odd number of constraints is violated simultaneously. Bounds on reliability can be determined from the probability that this function is nonnegative and from the probabilities of N_i.

$$\frac{1}{2}\left[1 + Pr\{\mathcal{F}(\boldsymbol{d}, \boldsymbol{p}) \geq 0\} - \sum_{i=1}^{n} Pr\{N_i\}\right] \geq R \geq Pr\{\mathcal{F}(\boldsymbol{d}, \boldsymbol{p}) \geq 0\} \tag{19}$$

The derivation of these bounds is given in the Appendix. These bounds consider some of the partial overlap of the N_i, so they are tighter than those given by Eq. 17. The distribution function for $\mathcal{F}(\boldsymbol{d}, \boldsymbol{p})$ is difficult to estimate because $\mathcal{F}(\boldsymbol{d}, \boldsymbol{p})$ is not monotonic. Errors in estimating this distribution function can result in significant errors in the bounds given by Eq. 19, so the simple bounds of Eq. 17 should always take precedence.

Estimating reliability of processes with operating degrees of freedom

Problems with no operating degrees of freedom are the exception rather than the rule in the chemical process industry, but they are easier to analyze. This section will consider methods of eliminating degrees of freedom. If all degrees of freedom can be eliminated, bounds on reliability can be determined from either Eq. 17 or Eq. 19.

In many processes, some constraints will always bound the feasible region regardless of the parameter values. To illustrate this point, consider absorption in a packed tower as shown in Figure 2. This process contains three constraints (composition of the exit gas, flooding, and pump capacity) and a single degree of freedom (liquid flow rate). The exit-gas composition constraint describes a lower bound on the liquid flow. The pump capacity describes a maximum-flow constraint, as does the flooding constraint. The relative positions of the constraints are illustrated in

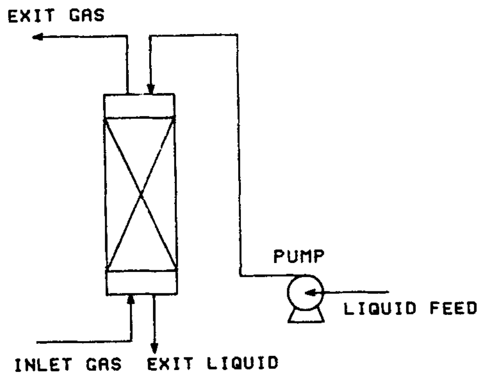

Figure 2. A simple absorption process.

Figure 3 for two realizations of the parameters. While the relative positions of the constraints change, the gas composition is always the lower bound on liquid flow.

If a constraint exists that always bounds the feasible region, the dimensionality of the problem can be reduced be considering only operating conditions that lie on the constraint boundary. This reduction is equivalent to setting the constraint to its limiting value and using it as a model equation to eliminate one degree of freedom. In the absorption example, this reduction corresponds to calculating the minimum liquid flow from the maximum exit gas composition. The design reliability problem simplifies to that of determining whether this flow violates the flooding or pump constraints.

If the degrees of freedom cannot be eliminated by the constraint structure, an alternate formulation of design reliability suggests a useful approximation. The solution to Eq. 4 yields α and $\boldsymbol{u}^*$, which is the vector of the "most reliable" operating conditions. The vector $\boldsymbol{u}^*$ is a function of both $\boldsymbol{d}$ and $\boldsymbol{p}$. If $\boldsymbol{u}^*(\boldsymbol{p})$ is known, all degrees of freedom have been eliminated and reliability can be calculated as follows.

$$R = Pr\{\max_i g_i[\boldsymbol{d}, \boldsymbol{u}^*(\boldsymbol{p}), \boldsymbol{p}] \leq 0\} \qquad (20)$$

Because $\boldsymbol{u}^*(\boldsymbol{p})$ is not known *a priori*, the reliability problem can be formulated as a variational problem.

$$R = \max_{\boldsymbol{u}^*(\boldsymbol{p})} Pr\{\max_i g_i[\boldsymbol{d}, \boldsymbol{u}^*(\boldsymbol{p}), \boldsymbol{p}] \geq 0\} \qquad (21)$$

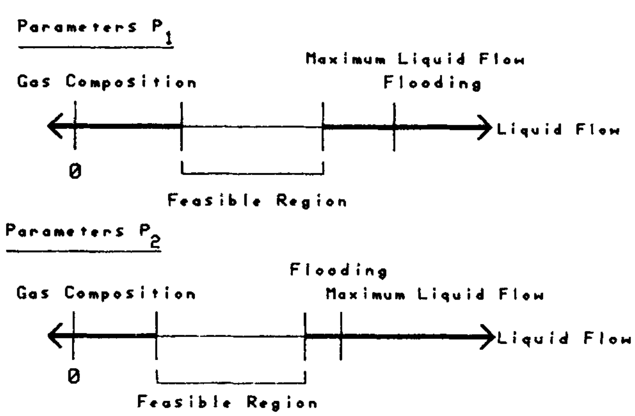

Figure 3. Relative constraint positions in absorption process.

While Eq. 21 is no easier to evaluate than Eq. 12, it suggests that design reliability can be estimated by estimating $\boldsymbol{u}^*(\boldsymbol{p})$. Equation 21 also implies that this estimated value of $\boldsymbol{u}^*(\boldsymbol{p})$ will yield a value of reliability that is a lower bound on the true value of reliability. The simplest approximation is to assume that the operating conditions are constant. A better estimate can be obtained by solving Eq. 4 for various perturbations of the uncertain parameters and approximating $\boldsymbol{u}^*$ by some simple function.

A consequence of Eq. 21 is that adding flexibility to a process can increase design reliability. Assuming $\boldsymbol{u}^*$ to be constant is equivalent to having no degrees of freedom. This approximation gives a lower bound on reliability. Relaxing this assumption or adding degrees of freedom can only increase reliability.

A procedure for estimating design reliability

Even with the simplification discussed in the previous section, the effort required to estimate design reliability can be prohibitive. A procedure is proposed in this section which enables the engineer to simplify the problem to a manageable size. This procedure is also designed to increase the engineer's understanding of the interaction of uncertainty and the design.

1. Define the Problem. To begin, all sources of uncertainty must be identified and quantified as probability distributions. It is equally important to identify all equipment constraints, process specifications, and operating degrees of freedom.

2. Eliminate Degrees of Freedom. To simplify the problem, specifications and constraints which bound the feasible region for all values of the uncertain parameters must be identified and used to eliminate operating degrees of freedom. In the absence of equipment turndown constraints, specifications usually represent lower bounds on process flows and utility consumptions and can hence be used to eliminate degrees of freedom.

3. Linear Analysis of Variance. All uncertainties and constraints are not equally important in estimating reliability. To minimize the computational effort, it is important to eliminate insignificant constraints and uncertainties. Linear analysis of variances is a useful tool for this step.

First, operating conditions are fixed at their nominal design values. The variance of each constraint can be estimated by a linear approximation.

$$\sigma_{g_i}^2 = \sum_{j=1}^{n} \left(\frac{\partial g_i}{\partial p_j}\right)^2_{p_{k,k\neq j}} \sigma_{p_j}^2 \qquad (22)$$

The relative contribution of parameter j to the total variance of constraint i is

$$r_{i,j} = \left(\frac{\partial g_i}{\partial p_j}\right)^2 \sigma_{p_j}^2/\sigma_{g_i}^2 \qquad (23)$$

The standard statistical distance of a constraint from its limiting value is defined as follows:

$$\delta_i = -g_i(\boldsymbol{d}, \boldsymbol{u}, \boldsymbol{p})/\sigma_{g_i} \qquad (24)$$

If the distance of a constraint from its limiting value is large, the likelihood of the constraint being violated is small, and it need not be considered in the analysis. A standard distance of 5 from the limiting value corresponds to a failure probability of less than 10^{-6} if the distribution function for g_i is Gaussian. Hence, constraints with δ_i greater than 5 can be ignored. If a particular parameter contributes little to the total uncertainty of all remaining constraints ($r_{i,j} < 0.03$ for all i), it can be considered to be deterministic.

4. *Operations Analysis.* If degrees of freedom remain in the process, functions must be postulated that relate operating conditions to the values of the uncertain parameters. The first step is to identify the "most reliable" operating conditions for the nominal values of the parameters. These operating conditions are obtained by solving Eq. 4 with the constraints scaled by the linear estimates of standard deviation. Equation 4 is then solved for various perturbations of the parameters. Finally, the results are represented by a simple algebraic equation.

5. *Calculation of Reliability.* Having simplified the problem and eliminated the degrees of freedom, bounds on reliability can be calculated using Eq. 19.

The process design engineer can obtain some insights to the interaction of the design and uncertainties from this procedure. First, the linear analysis of variance indicates the important uncertainties, and it can suggest experimental studies needed to reduce these uncertainties, which in turn reduce the required overdesign. Operations analysis indicates how flexibility interacts with the design uncertainties. Finally, in estimating overall uncertainties, the probability of each constraint being violated is also calculated. These probabilities indicate where a design can be modified to improve reliability and where a process has been excessively overdesigned.

Example

A design for separating benzene and toluene in a packed tower is used to illustrate the procedure for estimating design reliability. This distillation is to produce 99.5 mol % benzene and 99.5 mol % toluene from a feed with an average composition of 42 mol % benzene. A diagram of the process is given in Figure 4, and the details of the design are summarized in Table 1. The operating degrees of freedom are column pressure, degree of feed vaporization, cooling water flow rate, and steam flow rate to the reboiler.

A differential mass transfer model was used to analyze this tower. Constant relative volatility and constant molar overflow were assumed. The mass transfer coefficient was estimated using the correlation of Bolles and Fair (1979), and the flooding velocity was obtained from the Eckert correlation (Eckert, 1970). These correlations require surface tension and diffusivities. Surface tension was calculated using the parachor, liquid diffusivities were estimated using the Wilke-Chang equation, and vapor diffusivities were determined form Chapman-Enskog theory. All three of these procedures are treated by Reid et al. (1977).

Table 2 lists all of the uncertainties considered in this example. The standard deviations reflect typical variations. The constraints used to model the equipment capacities and the product specifications are given in Table 3. The fraction flooding is the ratio of the actual vapor velocity to the flooding velocity.

The constraints on product compositions represent the only lower bounds on steam flow to the reboiler and cooling water flow. Hence, the product composition constraints always bound the feasible region, and therefore they can be used to eliminate reboiler steam flow and cooling water flow as degrees of freedom. After this reduction, two degrees of freedom remain, column pressure and feed liquid fraction. The nominal values of column pressure and feed liquid fraction are 103 kPa and 1.0, respectively.

A linear analysis of variance was performed to identify the critical constraints and significant uncertainties. The relative contributions to variance and the standard distances to the limits are summarized in Table 4. Using the criterion given above in step 3 of the procedure, constraints having a nominal value with

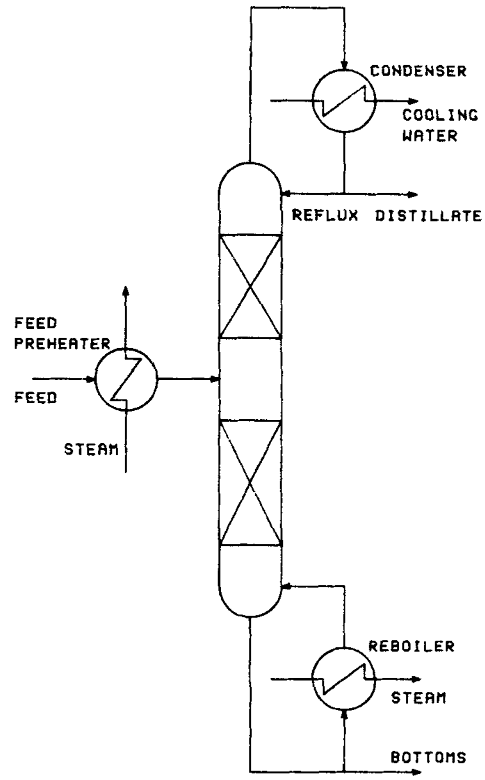

Figure 4. Flowsheet for benzene-toluene distillation example.

Table 1. Equipment Design for Benzene-Toluene Distillation Example

Feed Preheater	
Area	160 m^2
Max. steam press.	308 kPa
Max. steam flow	954 kg/h
Column	
Tower dia.	2.29 m
Packing type	Pall rings
Packing size	7.62 cm
Rectifying sect. hgt.	25.9 m
Stripping sect. hgt.	10.4 m
Max. press.	138 kPa
Condenser	
Area	487 m^2
Max. cooling water flow	95,500 kg/h
Reboiler	
Area	4,870 m^2
Max. steam press.	308 kPa
Max. steam flow	6,280 kg/h
Products	
Benzene comp.	99.5 mol %
Toluene comp.	99.5 mol %

Table 2. Uncertainties in Benzene-Toluene Distillation Example

Quantity	Type of Distribution	Mean	Std. Dev.
Relative volatility	Gamma	2.37	0.05
Mass trans. coeff.*	Normal	0.042	0.323
Flooding correlation*	Normal	−0.204	0.167
Feed comp., mol frac.	Beta	0.40	0.02
Feed rate, kmol/h	Uniform	591	32.7
Condenser U,** J/ $m^2 \cdot h \cdot K$	Gamma	3,071,000	327,000
Preheater U, J/ $m^2 \cdot h \cdot K$	Gamma	5,118,000	819,000
Reboiler U, J/ $m^2 \cdot h \cdot K$	Gamma	5,118,000	819,000
Cooling water temp., K	Normal	305.6	2.2
Liquid diffusivity*	Normal	0.0651	0.256
Surface tension*	Normal	0.032	0.054
Vapor diffusivity*	Normal	0.044	0.100

*Log ratio error in the correlation, $\epsilon = \ln (x_{actual}/x_{correlation})$.
**U = heat transfer coefficient.

a standard distance from the limit greater than 5 were eliminated. The constraints remaining after this reduction were reflux flow, reboiler steam pressure, and reboiler steam flow. Uncertainties that contribute less than 3% to the total variance of the remaining constraints were neglected. The significant uncertainties in this example were relative volatility, the mass transfer coefficient, the feed composition, the feed rate, and the reboiler heat transfer coefficient. Linear analysis of variance in this example enables a problem with nine constraints and 12 parameters to be reduced to one with three constraints and five parameters.

Equation 4 was solved for various perturbations of the significant uncertain parameters in order to estimate the "most reliable" operating policy. The variation of operating conditions with parameter values was obtained as well as the active constraints. The active constraints for nominal values of the parameters were the reboiler steam pressure constraint and the reboiler steam flow constraint. Hence, the amount of feed vaporization and the column pressure should be manipulated to avoid these constraints. The perturbation analysis of the operating conditions indicated that the column pressure should be fixed at its lower limit to avoid the reboiler steam pressure constraint and that the liquid fraction of the feed should be manipulated to compensate for uncertainty in the feed rate according to the following equation.

$$f_l = 0.56 + m_f/5{,}000 \quad (25)$$

The bounds on reliability were computed using Eq. 19. Probability distributions for the constraints were estimated using the quadratic approximation method described by Kubic (1986). Design reliability was computed for a fixed set of operating conditions and for the recommended operating policy. The results are summarized in Table 5. Note that operating flexibility increased the reliability, as predicted.

The desired reliability for an industrial process is 0.99. Because 0.99 is not achieved by this design, it must be modified to increase reliability. Table 5 indicates that the most probable causes of failure are insufficient reboiler steam flow and insufficient reflux flow. Modifications are required which decrease the probability of these failures. The linear analysis of variance indicates that the probability of failure by all other constraints except the reflux flow is small, which implies that there is excessive overdesign in some parts of the design.

Two alternate designs were considered in this study. In the first, the maximum reflux flow and maximum reboiler steam flow were increased by 10% in order to avoid these critical constraints. The second alternate design uses the same utility limits as the base design with a greater packing height. This method is the more traditional way of adding a safety factor. The details of the design changes are given in Table 6. The results of the reliability calculations are summarized in Table 7. Both alternatives increase the reliability to an acceptable level, indicating that there is more than one way to add a safety factor to a design. The decision of how to add a safety factor will depend on the process economics.

Table 3. Constraints for Benzene-Toluene Distillation Example

Constraint	Type	Limiting Value
Distillate comp.	Min.	0.995 mol frac. C_6H_6
Bottoms comp.	Max.	0.005 mol frac. C_6H_6
Cooling water flow	Max.	95,500 kg/h
Reflux flow	Max.	591 kmol/h
Rectifying sect. fract. flooding	Max.	1.0
Stripping sect. fract. flooding	Max.	1.0
Preheater steam press.	Max.	308 kPa
Preheater steam flow	Max.	954 kg/h
Reboiler steam press.	Max.	308 kPa
Reboiler steam flow	Max.	6,180 kg/h
Max. column press.	Max.	138 kPa
Min. column press.	Min.	101 kPa
Cooling water exit temp.	Max.	366.7 K

Design Reliability with Fuzzy Uncertainties

A fuzzy set is a set without sharp boundaries (Zadeh, 1965). Membership in a fuzzy set is described by a membership function. The membership function for fuzzy set A, $\mu_A(x)$, is a mapping from the universe X to the interval [0, 1]. The closer the membership function of x is to unity the more x belongs to A. A fuzzy set can be used to describe the possible realization of an uncertain parameter for which distinct bounds are not known.

Quantifying design confidence when the uncertainties are fuzzy requires an appropriate fuzzy measure. A possibility measure will be used in this study. A possibility measure, Π, is a fuzzy measure with the additional property that, for every A and B in $\mathcal{P}(X)$ such that $A \cap B = \emptyset$, $\Pi(A \cup B) = \max [\Pi(A), \Pi(B)]$ (Dubois and Prade, 1980). A possibility measure can be built from a possibility distribution (Zadeh, 1978), which is a function π from X to [0, 1] such that

$$\forall A, \Pi(A) = \sup_{x \in A} \pi(x) \quad (26)$$

Possibility measures the potential for an event, not the frequency of an event as probability does. As a simple example, consider the number of glasses of beer that graduate student R can drink in an evening. The possibility that R drinks a certain number of glasses depends on his capacity and not on the number he normally drinks. R's capacity is also not known with

Table 4. **Relative Contributions to Variance Determined by Linear Analysis for Benzene-Toluene Distillation Example**

Uncertain Parameters	Cooling Water Flow	Reflux Flow	Rectifying Sect. Flooding	Stripping Sect. Flooding	Preheater Steam Press.	Preheater Steam Flow	Reboiler Steam Press.	Reboiler Steam Flow	Cooling Water Exit Temp.
Relative volatility	0.29	0.44	0.14	0.16	0.10	0.01	0.16	0.36	0.21
Mass trans. coeff.	0.37	0.34	0.11	0.12	0	0	0.11	0.28	0.16
Flooding veloc.	0	0	0.70	0.66	0	0	0	0	0
Feed comp.	0.03	0	0	0	0.33	0.03	0.01	0.03	0.02
Feed rate	0.29	0.21	0.05	0.06	0.10	0.96	0.14	0.33	0.21
Condenser U*	0.02	0	0	0	0	0	0	0	0.36
Feed preheater U	0	0	0	0	0	0	0	0	0
Reboiler U	0	0	0	0	0	0	0.57	0	0
Cooling water temp.	0.15	0	0	0	0	0	0	0	0
Surface tension	0	0	0	0	0	0	0	0	0.04
Liquid diffusivity	0	0	0	0	0	0	0	0	0
Vapor diffusivity	0	0	0	0	0	0	0	0	0
Nominal value	59,200	434,000	0.454	0.492	111	225	264	5,360	351.1
Approx. std. dev.	6,290	38,200	0.082	0.092	0.51	13	13.1	404	1.2
Std. dist. to limit	5.77	4.11	6.58	5.52	38.6	56.1	3.36	2.03	13.4

*U = heat transfer coefficient

absolute certainty because it depends on the size of the glass and on what R ate for supper. Thus, the possibility that R can drink one glass of beer is close to unity because most people can drink one glass unless it it extremely large. However, the possibility that he can drink 30 glasses of beer is near zero. Between one glass and 30 glasses, there is a gradual transition from a definite possibility to no possibility.

This simple example can also be used to illustrate the difference between possibility and probability. During 90% of his evenings, R drinks one glass of beer. Because he drinks one glass on a regular basis, the possibility of drinking one glass is essentially unity. Also, R frequently drinks one glass; therefore, the probability that he drinks one glass on a given day is large. Although he can definitely drink two glasses, he rarely does. Drinking two glasses of beer is possible but not probable. This example illustrates that an event that is probable is also possible, but that possibility does not imply probability.

To apply these ideas to process design, it should be noted that a fuzzy set of parameter values describes the possible values that the parameter can assume. Measuring design confidence involves determining the possibility that there are realizations of the parameters for which the design works and realizations of the parameters for which the design fails. Fuzzy reliability is therefore defined as the possibility that M is not empty, and the design works.

$$\mathcal{R} = \Pi(M) \tag{27}$$

Likewise, fuzzy unreliability is the possibility that a design fails.

$$\mathcal{U} = \Pi(N) \tag{28}$$

Fuzzy reliability and unreliability can be expressed in terms of the function $\alpha(\boldsymbol{d}, \boldsymbol{p})$.

$$\mathcal{R} = \Pi[\alpha(\boldsymbol{d}, \boldsymbol{p}) \leq 0] \tag{29}$$

$$\mathcal{U} = \Pi[\alpha(\boldsymbol{d}, \boldsymbol{p}) > 0] \tag{30}$$

The membership function for $\alpha(\boldsymbol{d}, \boldsymbol{p})$ can be determined using the extension principle (Zadeh, 1975).

$$\text{s.t.} \quad \mu_\alpha(\alpha) = \sup_{\substack{\boldsymbol{p} \\ \alpha = \alpha(\boldsymbol{d}, \boldsymbol{p})}} \min_i [\mu_{p_i}(p_i)] \tag{31}$$

Using the definition of the possibility distribution, fuzzy reliability and unreliability are expressed as follows.

$$\text{s.t.} \quad \mathcal{R} = \sup_{\substack{\boldsymbol{p} \\ \alpha(\boldsymbol{d}, \boldsymbol{p}) \leq 0}} \min_i [\mu_{p_i}(p_i)] \tag{32}$$

Table 5. **Reliability of Base Benzene-Toluene Distillation Design**

	Fixed Operations	Variable Operations
Column pressure, kPa	101	101
Feed liquid frac.	1.0	Varies
Upper bound on reliability	0.9308	0.9668
Lower bound on reliability	0.9006	0.9265
Constraint Violation Probabilities		
Reflux flow	0.0020	0.0350
Reboiler steam press.	0.0283	0.0050
Reboiler steam flow	0.0691	0.0332

Table 6. **Alternate Designs for Benzene-Toluene Distillation Example**

	Base Design	Alt. I	Alt. II
Rectifying sect. hgt., m	25.9	25.9	32.3
Stripping sect. hgt., m	10.4	10.4	12.8
Max. reflux flow, kmol/h	591	650	591
Max. reboiler steam flow, kg/h	6,180	6,800	6,180

Table 7. Reliability of Alternate Benzene-Toluene Distillation Designs

	Base Design	Alt. I	Alt. II
Upper bound on reliability	0.9668	0.9950	0.9907
Lower bound on reliability	0.9265	0.9917	0.9865
Constraint Violation Probabilities			
Reflux flow	0.0350	0.0012	0.0021
Reboiler steam press.	0.0050	0.0050	0.0020
Reboiler steam flow	0.0332	0.0020	0.0093
Expected Values for Flows			
Cooling water, kg/h	61,950	61,950	61,050
Reboiler steam, kg/h	5,109	5,109	4,995

$$\mathcal{U} = \sup_{\substack{p \\ \text{s.t. } \alpha(d,p)>0}} \min_i \left[\mu_{p_i}(p_i)\right] \tag{33}$$

Because sets M and N are disjoint, the possibility of their union is given by the definition of possibility.

$$\Pi(M \cup N) = \max(\mathcal{R}, \mathcal{U}) \tag{34}$$

This possibility can be expressed in terms of the membership functions of the fuzzy parameters.

$$\Pi(M \cup N) = \sup_p \min_i \left[\mu_{p_i}(p_i)\right] = \text{hgt}\,(p_1 \times \cdots \times p_r) \tag{35}$$

The height of a fuzzy set, hgt $(\cdot)$, is the minimum upper bound on the membership function. Because the possibility of $M \cup N$ is an upper bound on fuzzy reliability and the height of each fuzzy parameter is an upper bound on the possibility of $M \cup N$, it follows that

$$\text{hgt}\,(p_i) \geq \mathcal{R} \tag{36}$$

and

$$\min_i \text{hgt}\,(p_i) \geq \mathcal{R} \tag{37}$$

While a possibilistic formulation of design reliability parallels the probabilistic formulation, the results are different and require some explanation. Fuzzy design reliability measures the possibility that there is some realization of the parameters in the fuzzy set of parameters for which the design works. Likewise, fuzzy unreliability is the possibility that there is some realization of the parameters for which the design fails. A good design maximizes reliability and minimizes unreliability. Unlike the probabilistic formulation, a large value for fuzzy reliability does not imply a small value for fuzzy unreliability. If both fuzzy reliability and unreliability are large, the design may work, but it may not be able to compensate for all of the uncertainty. Further study is required to determine whether the design is acceptable. If the uncertainty is due to the expected variation in feeds or other operating conditions, large reliability and large unreliability imply that the design is not flexible enough. Fuzzy parameters can also represent model uncertainty (Kubic and Stein, 1986). Large reliability and large unreliability for designs subject to model uncertainty indicate that the model is not precise enough to evaluate the design, so the design must be modified to accommodate the uncertainty or a better model and/or experiments are needed to verify the design.

Equation 34 is analogous to stochastic reliability and unreliability summing to unity, but there is an important difference. The righthand side of Eq. 34 can be less than unity. If it is less than unity, all possibilities have not been considered. The maximum value of fuzzy reliability is related to the height of the fuzzy parameters. If the height of a fuzzy parameter is less than unity, knowledge of that parameter is not complete. In the methods given by Kubic and Stein (1986) for evaluating model uncertainty, a fuzzy set of errors with a height less than unity implies that there is insufficient data for evaluating the model. If no data exist for testing a model, no statements can be made about its accuracy, and hence no statements can be made about the reliability or unreliability of a design based on that model. This lack of knowledge is termed ignorance, and the ignorance factor is defined as follows:

$$I = 1 - \text{hgt}\,(p_1 \times \cdots \times p_r) \tag{38}$$

A large value of the ignorance factor shows a need for information or experimental data. Because ignorance depends only on the fuzzy parameters, the need for data can be evaluated independently of the design. The authors recommend that the ignorance factor be less than 0.3 for an acceptable design.

Evaluating fuzzy design reliability

Reliability is calculated using Eq. 32. This equation is a nonlinear sup-min problem subject to a nonlinear constraint. To complicate the problem further, the constraint involves $\alpha(\boldsymbol{d}, \boldsymbol{p})$, which is defined as the solution to a constrained inf-max problem, Eq. 4.

The constraint in Eq. 32 restricts the parameters to values that simultaneously satisfy all constraints. The same region of parameter space could be described directly by the constraints, which eliminates the need to solve Eq. 4, but this formulation introduces a vector of operating conditions into the problem. For the special case of fixed operating conditions, the constraint on α can simply be replaced by the process contraints.

$$\mathcal{R}^*(\boldsymbol{u}) = \sup_{\substack{p \\ \text{s.t. } g_j(d,u,p)\leq 0}} \min_i \left[\mu_{p_i}(p_i)\right] \tag{39}$$

If operating conditions can vary, $\mathcal{R}^*(\boldsymbol{u})$ is a lower bound on fuzzy reliability because a set of operating conditions may exist that gives a greater value for reliability. Hence, the actual fuzzy reliability can be evaluated by maximizing $\mathcal{R}^*$ with respect to the operating conditions.

$$\mathcal{R} = \max_u \sup_{\substack{p \\ \text{s.t. } g_j(d,u,p)\leq 0}} \min_i \left[\mu_{p_i}(p_i)\right] \tag{40}$$

The minimization with respect to the index i can be transformed into a constrained maximization with respect to a new continuous variable, β.

$$\min_i \mu_{p_i}(p_i) = \max_{\substack{\beta \\ \text{s.t. } \mu_{p_i}(p_i)-\beta\geq 0}} \beta \tag{41}$$

Fuzzy reliability can now be calculated as follows:

$$\mathcal{R} = \sup_{\substack{u,p,\beta \\ \text{s.t.} \ \mu_{p_i}(p_i)-\beta\geq 0 \\ g_j(d,u,p)\leq 0}} \beta \tag{42}$$

This expression is a standard nonlinear programming problem that can be solved by any appropriate method. Successive linear programming works well if the $\mu_{p_i}(p_i)$ are convex functions.

Evaluating fuzzy design unreliability

Fuzzy unreliability, like fuzzy reliability, is determined by solving a constrained, nonlinear sup-min problem, Eq. 33. The constraint in this problem $[\alpha(d, p) > 0]$ describes the condition that at least one constraint is violated. The condition of at least one, however, cannot be reduced to a set of continuous constraints. Hence, Eq. 33 cannot be converted to a standard nonlinear programming problem.

In order to simplify the problem to a manageable form, consider the special case of fixed operating conditions. This case is important because it is an upper bound on the actual unreliability. It is also useful to consider only one constraint at a time. The set of parameters corresponding to design failure is the union of the sets of parameters that violate each constraint. By the axioms of possibility theory, the possibility of design failure is the maximum of individual constraint violation possibilities. The possibility of violating constraint j for a fixed set of operating conditions is calculated as follows:

$$\mathcal{U}_j^*(u) = \sup_{\substack{p \\ \text{s.t.} \ g_j(d,u,p)>0}} \min_i \ [\mu_{p_i}(p_i)] \tag{43}$$

It follows that the system unreliability for a fixed set of operating conditions is

$$\mathcal{U}^*(u) = \max_j \{\sup_{\substack{p \\ \text{s.t.} \ g_j(d,u,p)>0}} \min_i \ [\mu_{p_i}(p_i)]\} \tag{44}$$

Minimizing $\mathcal{U}^*(u)$ with respect to the operating conditions will yield an upper bound that is a better estimate of unreliability than Eq. 44.

$$\mathcal{U} \leq \inf_u (\max_j \{\sup_{\substack{p \\ \text{s.t.} \ g_j(d,u,p)>0}} \min_i \ [\mu_{p_i}(p_i)]\}) \tag{45}$$

If unreliability is low, as is desired, this upper bound should be fairly close to the actual unreliability. Equation 45 will also be a good estimate of unreliability if operating conditions are not sensitive to the fuzzy parameters.

Although Eq. 45 cannot be converted into a standard nonlinear programming problem, it does suggest a hierarchy of calculations. Assuming a set of operating conditions, the failure possibility for each constraint can be calculated by transforming Eq. 43 into a standard constrained maximization.

$$\mathcal{U}_j^*(u) = \sup_{\substack{p,\sigma \\ \text{s.t.} \ \mu_{p_i}(p_i)-\sigma\geq 0 \\ g_j(d,u,p)>0}} \sigma \tag{46}$$

The maximum $\mathcal{U}_j^*(u)$ can be easily found by enumeration to give $\mathcal{U}^*(u)$. The problem now is to find an efficient algorithm to maximize $\mathcal{U}_j^*(u)$ with respect to the operating conditions.

A heuristic procedure was developed to solve this maximization. Intuitively, unreliability should be minimized if the operating conditions maximize the "distance" from the constraints. These operating conditions can be determined by solving Eq. 4 for scaled values of the constraints. The min-max formulation of fuzzy set theory suggests the following scalings.

$$\delta_j = \max \left| \left(\frac{\partial g_j}{\partial p_i}\right)(p_{i,max} - p_{i,min}) \right| \tag{47}$$

Using this heuristic, the following successive approximation can be used to estimate the upper bound on fuzzy unreliability.

1. Set the parameters to their nominal values.
2. Solve Eq. 4 using the current values of the parameters and the scalings given by Eq. 47 to obtain an estimate of the best operating conditions.
3. Check convergence by comparing the operating conditions found in step 2 to the previous estimate. If they are equal, terminate the algorithm. This step is skipped on the first iteration.
4. Calculate unreliability and the corresponding parameter values from Eqs. 44 and 46.
5. Set the assured parameters equal to the values obtained in step 4 and return to step 2.

Because this algorithm is a heuristic, convergence and stability are not guaranteed. The algorithm was found to give reasonable estimates of unreliability in a few iterations for the problems on which it was tested, but the operating conditions were not very sensitive to the fuzzy parameters.

By Eqs. 34 and 35, the maximum of the fuzzy reliability and unreliability is the height of the fuzzy set of parameters. This relation implies that if fuzzy unreliability is less than the height

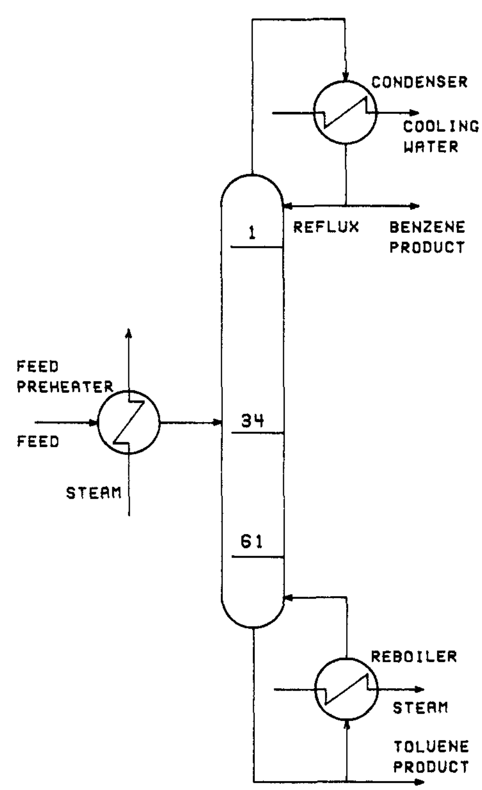

Figure 5. Flowsheet for benzene-toluene distillation example.

Table 8. Equipment Design for Benzene-Toluene Distillation Example

Feed Preheater	
Area	39.5 m^2
Max. steam press.	310 kPa
Max. steam flow	3,320 kg/h
Column	
Tray type	Ballast trays
No. trays	61
Feed tray location	34
Max. reflux flow	1,000 kmol/h
Tower dia.	2.77 m
Tray spacing	0.61 m
Press. drop	14 kPa
Max. press.	138 kPa
Condenser	
Area	836 m^2
Max. cooling water flow	218,200 kg/h
Cooling water temp.	305.6 K
Reboiler	
Area	353 m^2
Max. steam press.	310 kPa
Max. steam flow	12,720 kg/h
Products	
Benzene comp.	99.5 mol %
Toluene comp.	99.5 mol %

of the set of parameters, the reliability must equal the height. Because the height is easily determined and because reliability is more difficult to calculate than unreliability, unreliability should be calculated first. If the upper bound on unreliability is less than the height, reliability equals the height and there is no need to solve Eq. 42.

Example

A design of a distillation column for separating a benzene-toluene mixture into 99.5 mol % benzene and 99.5 mol % toluene is used to illustrate the calculation of fuzzy reliability and unreliability. A diagram of the process is given in Figure 5, and the details of the design are summarized in Table 8. The feed conditions are subject to uncertainty and are given later. This particular design contains four operating degrees of freedom: the preheater exit temperature, the condenser pressure, the cooling water flow rate, and the reboiler steam flow rate.

Constant molar overflow was assumed in the analysis of the tower. Regular solution theory was used to estimate activity coefficients, and the vapor was assumed to be an ideal gas. Liquid volumes were estimated using a linear mixing rule. Physical

Table 9. Nominal Parameter Values for Benzene-Toluene Distillation Example

Physical Properties	
Benzene solubility parameter	18.8 $(J/mL)^{1/2}$
Benzene molec. wt.	78.11 g/gmol
Benzene molar vol.	88.3 mL/gmol
Toluene solubility parameter	18.2 $(J/mL)^{1/2}$
Toluene molec. wt.	92.14 g/gmol
Toluene molar vol.	106.33 mL/gmol
Transport Coefficients	
Preheater heat trans. coeff.	5,120,000 $J/m^2 \cdot h \cdot K$
Condenser heat trans. coeff.	3,080,000 $J/m^2 \cdot h \cdot K$
Reboiler heat trans. coeff.	5,120,000 $J/m^2 \cdot h \cdot K$
Tray efficiency	57%

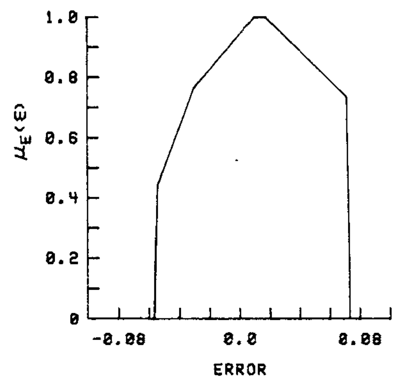

Figure 6. Fuzzy set of errors for regular solution theory applied to benzene-toluene system.

properties are given in Table 9. Flooding velocities were estimated using the correlation recommended by Glitsch (1984).

Four fuzzy uncertainties were considered: uncertainty in regular solution theory, uncertainty in tray efficiency, uncertainty in feed rate, and uncertainty in feed composition. The error in regular solution theory was expressed as a binary interaction parameter.

$$RT \ln \gamma_k = v_k \Phi_j^2 [(\delta_k - \delta_j)^2 + 2\epsilon\delta_k\delta_j] \tag{48}$$

where

$$\Phi_j = \frac{v_j x_j}{v_k x_k + v_j x_j}$$

The fuzzy set of possible errors for regular solution theory was estimated using the methods of Kubic and Stein (1986) and is shown in Figure 6. The membership functions for the other fuzzy uncertainties are approximations used to illustrate the methods. These membership functions are shown in Figures 7 through 9.

The constraints used to model the equipment capacities and the product specifications are given in Table 10. The fraction of flooding velocity is the ratio of the actual vapor velocity to the flooding velocity. The constraints on product compositions represent the only lower bounds on the steam flow to the reboiler and cooling water flow. As in the probabilistic example, the product composition constraints always bound the feasible re-

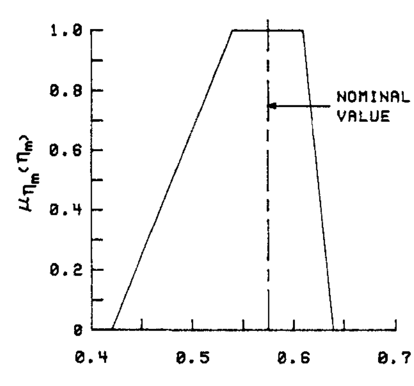

Figure 7. Membership function for tray efficiency.

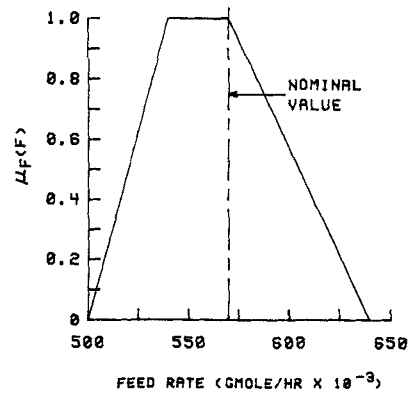

Figure 8. Membership function for feed rate.

gion; therefore, they can be used to eliminate reboiler steam flow and cooling water flow as degrees of freedom. After this reduction, two degrees of freedom remain, column pressure and preheater exit temperature.

The upper bound on fuzzy unreliability was estimated using the procedure given in the previous section. The upper bound of 0.323 for unreliability is less than the height of the fuzzy set of parameters, which is unity in this example; hence, fuzzy reliability is simply unity. In addition to reliability and unreliability, constraint failure possibilities were obtained from the analysis along with the values of the operating conditions that minimize the possibility of failure. The results of the analysis are summarized in Table 11.

Because fuzzy reliability equals the height of the set of fuzzy parameters, the design will work for the nominal values of the parameters. This conclusion is predicted because the design was based on nominal parameter values. The nonzero value of fuzzy unreliability indicates that there is a possibility that the design may fail, but the question is whether 0.323 is a significant possibility of failure. The authors' limited experience with fuzzy uncertainties indicates that events with possibilities of less than 0.5 are not likely to occur. Hence, a maximum unreliability of 0.3 is a reasonable limit for a good design. This unreliability criterion is satisfied for this example, so the design is acceptable.

The constraint failure possibilities indicate that failure is most likely to be caused by insufficient reflux capacity. If unreliability is to be reduced, the design variables that affect the reflex constraint must be changed.

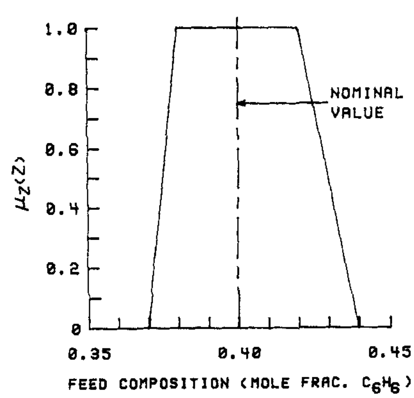

Figure 9. Membership function for feed composition.

Table 10. Constraints for Benzene-Toluene Distillation Example

Constraint	Limiting Value
Feed preheater steam press.	310 kPa
Feed preheater steam flow	3,320 kg/h
Min. preheater exit temp.	Sat. liquid
Cooling water flow rate	218,200 kg/h
Reflux rate	1,000 kmol/h
Frac. flooding veloc.	1.00
Reboiler steam press.	310 kPa
Reboiler steam flow	12,720 kg/h
Max. column press.	138 kPa
Min. column press.	101 kPa
Max. toluene in distillate	0.5 mol %
Max. benzene in bottoms	0.5 mol %

Design Reliability with Stochastic and Fuzzy Uncertainties

In the previous sections, uncertainties were considered to be either stochastic or fuzzy. Real process designs are often subject to both types of uncertainties simultaneously, so it is necessary to have a measure of design confidence that considers all uncertainties. The axioms of fuzzy set theory, however, are not compatible with probability theory; so a simple fuzzy measure is not appropriate.

The concept of the probability of a fuzzy event (Dubois and Prade, 1980) can be used to address this problem. Let A be a fuzzy set in the universe X, and $p(x)$ be the probability density function for a random variable x. The probability of A is defined as follows (Dubois and Prade):

$$Pr\{A\} = \int_X \mu_A(x)\, p(x)\, dx \tag{49}$$

To apply this concept to design reliability, the vector of parameters, $\boldsymbol{p}$ is partitioned into a vector of stochastic parameters, $\boldsymbol{p}_s$, and a vector of fuzzy parameters, $\boldsymbol{p}_f$. For a given realization of the stochastic parameters, fuzzy reliability and unreliability can be evaluated.

$$\mathcal{R}(\boldsymbol{d}, \boldsymbol{p}_s) = \sup_{\substack{\boldsymbol{p}_f \\ \text{s.t. } \alpha(\boldsymbol{d}, \boldsymbol{p}_s, \boldsymbol{p}_f) \le 0}} \min_i \; \mu_{p_{f,i}}(p_{f,i}) \tag{50}$$

$$\mathcal{U}(\boldsymbol{d}, \boldsymbol{p}_s) = \sup_{\substack{\boldsymbol{p}_f \\ \text{s.t. } \alpha(\boldsymbol{d}, \boldsymbol{p}_s, \boldsymbol{p}_f) > 0}} \min_i \; \mu_{p_{f,i}}(p_{f,i}) \tag{51}$$

Table 11. Fuzzy reliability and Unreliability for Benzene-Toluene Distillation Example

Fuzzy reliability	1.000
Upper bound on fuzzy unreliability	0.323
Constraint Failure Possibility	
Preheater steam press.	0.188
Preheater steam flow	0
Flooding	0
Reflux flow rate	0.323
Reboiler steam press.	0
Reboiler steam flow	0.043
Condenser cooling water flow	0.058
Minimum Failure Operating Conditions	
Feed preheater exit temp., K	371.7
Condenser press., kPa	103

Fuzzy reliability and fuzzy unreliability can be used to define fuzzy sets of stochastic parameters for which the design will work and for which it will fail.

$$\mathbf{R} = \{[\boldsymbol{p}_s, \mu_R(\boldsymbol{p}_s)] \mid \boldsymbol{p}_s \in \boldsymbol{P}_s, \mu_R(\boldsymbol{p}_s) = \mathcal{R}(\boldsymbol{d}, \boldsymbol{p}_s)\} \tag{52}$$

$$\mathbf{U} = \{[\boldsymbol{p}_s, \mu_U(\boldsymbol{p}_s)] \mid \boldsymbol{p}_s \in \boldsymbol{P}_s, \mu_U(\boldsymbol{p}_s) = \mathcal{U}(\boldsymbol{d}, \boldsymbol{p}_s)\} \tag{53}$$

$\boldsymbol{R}$ and $\boldsymbol{U}$ are fuzzy events corresponding to the success and failure of the design. Design reliability and unreliability are the probabilities of these fuzzy events.

$$R(\boldsymbol{d}) = Pr\,\{\boldsymbol{R}\} \tag{54}$$

$$U(\boldsymbol{d}) = Pr\,\{\boldsymbol{U}\} \tag{55}$$

These probabilities can be expressed by the following integrals.

$$R(\boldsymbol{d}) = \int_{\boldsymbol{P}_s} \mathcal{R}(\boldsymbol{d}, \boldsymbol{p}_s) p(\boldsymbol{p}_s)\, d\boldsymbol{p}_s \tag{56}$$

$$U(\boldsymbol{d}) = \int_{\boldsymbol{P}_s} \mathcal{U}(\boldsymbol{d}, \boldsymbol{p}_s) p(\boldsymbol{p}_s)\, d\boldsymbol{p}_s \tag{57}$$

Note that reliability and unreliability are essentially the average fuzzy reliability and fuzzy unreliability.

In the special case of stochastic uncertainty only, $\mathcal{R}$ and $\mathcal{U}$ are restricted to values of zero and unity, and $\boldsymbol{R}$ and $\boldsymbol{U}$ are ordinary, complementary sets. R and U are ordinary probabilities, in this case corresponding respectively to the probability that the design will work and that the design will fail. Hence, these equations reduce to stochastic reliability. The other limiting case is that of fuzzy uncertainty only. For this case, $p(\boldsymbol{p}_s)$ is the Dirac delta function; so the general expressions reduce to fuzzy reliability and fuzzy unreliability as described in the previous section.

The membership function of $\boldsymbol{R} \cup \boldsymbol{U}$ is equal to the height of the fuzzy set of parameters for all values of the stochastic parameters.

$$\mu_{R \cup U}(\boldsymbol{p}_s) = \text{hgt}\,(p_{f,1} \times \cdots \times p_{f,r}) \tag{58}$$

The membership function of $\boldsymbol{R} \cup \boldsymbol{U}$ is independent of the stochastic parameters; therefore, the probability of $\boldsymbol{R} \cup \boldsymbol{U}$ is simply the height of the fuzzy set of parameters. Deviation of this quantity from unity indicates a lack of information about the possible values of the fuzzy parameters. Hence, an ignorance factor can be defined for the general problem.

$$I = 1 - \text{hgt}\,(p_{f,1} \times \cdots \times p_{f,r}) \tag{59}$$

This definition is identical to Eq. 38; it measures the need for information about the fuzzy uncertainties.

Computing reliability and unreliability

Calculating design reliability and unreliability involves evaluating the integrals given by Eqs. 56 and 57. Because of their complexity, Monte Carlo is the only practical method of evaluating these integrals. A major problem with Monte Carlo methods is that they can require extensive computational effort. Hence, it is important to simplify the problem as much as possible. This section outlines a procedure for simplifying the problem and estimating reliability.

1. Before begining the calculations, all possible equipment constraints and process specifications must be determined. Operating degrees of freedom must also be identified.

2. All sources of uncertainty must be identified and classified as either stochastic or fuzzy. They must also be quantified. Probability distributions for random errors can be determined from statistics. Fuzzy uncertainties can be determined using the procedures described by Kubic and Stein (1986).

3. To simplify the problem, specifications and constraints that bound the feasible operating region for all values of the uncertain parameters can be used to eliminate operating degrees of freedom. Such inequality constraints are set to equality constraints, which enables some of the operating conditions to be calculated. This procedure reduces the dimensionality of the problem and saves computation time.

4. Linear analysis is useful for identifying the significant constraints and uncertainties. The min-max formulation of fuzzy set theory suggests the following measure of the relative importance of each uncertainty on a particular constraint.

$$r_{i,j} = \left| \frac{\partial g_i}{\partial p_j} \right| \Delta p_j / \max_k \left| \frac{\partial g_i}{\partial p_k} \right| \Delta p_k \tag{60}$$

The derivatives are evaluated at nominal values of the operating conditions and the parameters. For fuzzy parameters, Δp_j is the maximum possible variation in the parameter p_j. For stochastic parameters, Δp_i is taken to be four times the standard deviation. If $r_{i,j}$ is small for a particular uncertainty in all constraints, that uncertainty can be neglected. A limiting value of 0.1 is recommended.

A distance based on the maximum possible variation can be defined to identify the important constraints.

$$d_i = |g_i| \Big/ \sum_j \left| \frac{\partial g_i}{\partial p_j} \right| \Delta p_j \tag{61}$$

A large value for d_i indicates that constraint i is not likely to be violated.

5. Design reliability and unreliability are estimated by generating random sets of the vector of stochastic parameters. For a given realization of the vector of stochastic parameters, fuzzy reliability and the upper bound on fuzzy unreliability can be calculated using the methods of the previous section. Finally, the integrals given by Eqs. 56 and 57 can be estimated by averaging the reliability and unreliability of the total sample. It should be noted that in estimating fuzzy reliability and unreliability, constraint violation possibilities are also estimated; so average constraint failure possibilities can be easily obtained.

Using design reliability and unreliability

The probability of a fuzzy event, as applied to design reliability, is not a probability in the ordinary sense. The general definitions of design reliability and unreliability are neither purely stochastic nor purely fuzzy. The degree of randomness or fuzziness depends on the relative magnitudes of the stochastic uncertainties and the fuzzy uncertainties. A simple test of the degree of fuzziness is to compare the maximum constraint failure possibility with the unreliability. If the two are nearly equal, the uncertainty is primarily fuzzy. Another test is to check the sum of reliability and unreliability. If the sum is nearly unity, the uncertainty is primarily random. A final test is to examine the results of the linear analysis. If the relative importance of the fuzzy uncertainties is much greater than the stochastic uncertainties, reliability and unreliability are primarily fuzzy. Reliability and unreliability are primarily stochastic if the opposite is true.

Unless the uncertainty is primarily random, design reliability and unreliability should be treated as the potential of success and failure and not as a frequency. According to Zadeh (1978), events that are probable are also possible, but possibility does not imply probability. This interpretation of possibility implies that the set of probable events is a subset of the set of possible events. Therefore, interpreting both reliability and unreliability as possibilities is always conservative because a larger set of parameters is considered.

The criteria for an acceptable design should be consistent with the criteria for a design subject only to fuzzy uncertainties, and it should also be consistent with the criteria for a design subject only to stochastic uncertainties. The ignorance factor depends only on the fuzzy parameters, so it should be less than 0.3 as recommended for problems with only fuzzy uncertainties. Fuzzy parameter values corresponding to the height of the fuzzy set represent the nominal values. If design reliability equals the height, the design can compensate for all of the stochastic uncertainty. Hence, the ratio of reliability to the height of the fuzzy set of parameters measures the ability of the design to compensate for the stochastic uncertainties. Requiring this ratio to be greater than 0.99 insures that the probability of success for a design subject only to stochastic uncertainties is greater than 99%. Limiting unreliability to a maximum value of 0.3 insures that the failure possibility of a design subject only to fuzzy uncertainties is less than 0.3.

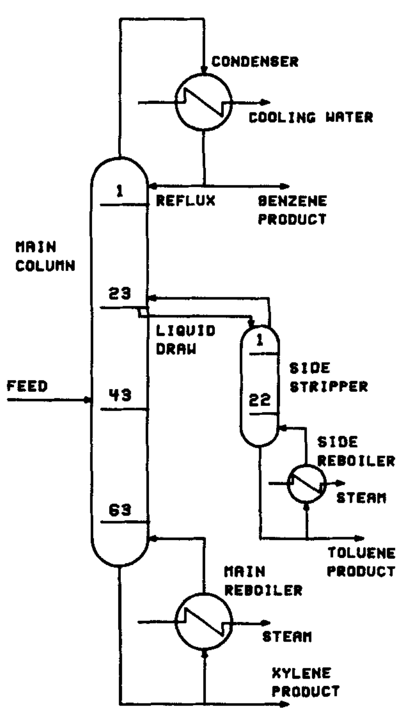

Figure 10. Benzene–toluene–*m*-xylene separation process.

Example

This example considers the separation of a benzene–toluene–*m*-xylene mixture using a distillation scheme proposed by Alatiqi (1985). The process, as illustrated in Figure 10, consists of a main column and a side stripper. The benzene and xylene products are produced in the main column, and the toluene is purified in the side stripper. The feed to the process is a saturated liquid with uncertainty in the feed composition and flow rate. Nominally, the feed rate is 273 kmol/h with a composition of 45 mol % benzene, 10 mol % toluene, and 45 mol % *m*-xylene. Process specifications are satisfied by manipulating four variables: the cooling water flow, the sidestream draw rate, the steam flow to the main reboiler, and the steam flow to the side reboiler. Pressure in the column is fixed at 202.7 kPa. The details of the design are given in Table 12.

The constraints on this process include the standard flooding limits, utility limits, pumping capacities, and product specifications. In addition to these constraints, the cooling water exit temperature cannot exceed its boiling point. Failure due to this constraint is a real possibility because the condenser temperature is above 373 K. A list of the constraints is given in Table 13.

The process was rigorously modeled using the ASPEN-PLUS program (Aspen Technology, 1981). Vapor-liquid equilibrium was estimated using regular solution theory for the liquid and the ideal-gas law for the vapor. Liquid volumes were calculated using a linear mixing rule.

Table 12. Equipment Design Parameters and Product Specifications for Benzene-Toluene–*m*-Xylene Separation Example

Main Column	
No. trays	63
Feed tray	43
Side draw tray	23
Type of tray	Ballast
Column dia.	1.65 m
Tray spacing	0.610 m
Reflux capacity	318 kmol/h
Side Stripper	
No. trays	22
Type of tray	Ballast
Column dia.	0.671 m
Tray spacing	0.457 m
Liquid draw capacity	86.4 kmol/h
Condenser	
Area	83.6 m^2
Max. cooling water flow	90,900 kg/h
Press.	202.7 kPa
Main Reboiler	
Area	325 m^2
Max. steam press.	1,034 kPa
Max. steam flow	6,360 kg/h
Side Reboiler	
Area	11.6 m^2
Max. steam press.	1,034 kPa
Max. steam flow	841 kg/h
Product Specifications	
Benzene comp.	95 mol %
Toluene comp.	90 mol %
Xylene comp.	95 mol %

Table 13. Constraints for Benzene–Toluene–*m*-Xylene Separation Example

Constraint	Limiting Value
Cooling water flow rate	90,900 kg/h
Cooling water exit temp.	373 K
Reflux rate	318 kmol/h
Side draw flow	84.6 kmol/h
Frac. flooding in main column	1.0
Main reboiler steam press.	1,034 kPa
Main reboiler steam flow	6,360 kg/h
Frac. flooding in side stripper	1.0
Side reboiler steam press.	1,034 kPa
Side reboiler steam flow	841 kg/h
Benzene product comp.	95 mol %
Toluene product comp.	90 mol %
Xylene product comp.	95 mol %

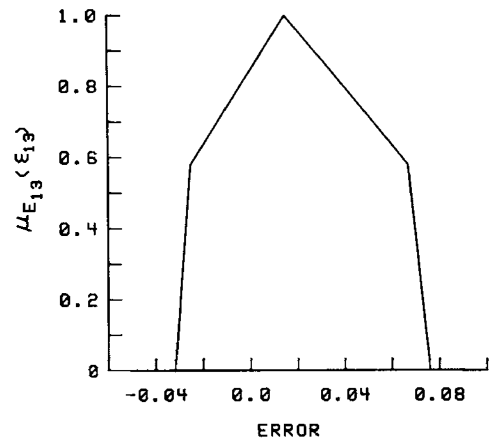

Figure 12. Membership function for error in benzene–*m*-xylene pair.

This design is subject to both fuzzy and stochastic uncertainties. The principle sources of fuzzy uncertainty are errors in the equilibrium model. Uncertainties in the tray efficiency correlation and in the feed are also considered to be fuzzy. All other uncertainties are stochastic; they include heat transfer coefficients, flooding velocities, and cooling water temperature.

The uncertainties in regular solution theory are expressed as binary interaction parameters.

$$RT \ln \gamma_k = v_k \sum_{i=1}^{n} \sum_{j=1}^{n} \Phi_i \Phi_j \left[D_{ik} - \frac{1}{2} D_{ij} \right] \tag{62}$$

where

$$\Phi_i = v_i x_i \Big/ \sum_{j=1}^{n} v_j x_j$$

$$D_{ij} = (\delta_i - \delta_j)^2 + 2\epsilon_{ij}\delta_i\delta_j \qquad (\epsilon_{ij} = \epsilon_{ji} \text{ and } \epsilon_{ii} = 0)$$

The ϵ_{ij} are the error parameters. Membership functions for the error parameters were evaluated using the methods of Kubic and Stein (1986); they are plotted in Figures 11 through 13.

Uncertainty in the feed is expressed as uncertainty in the component flow rates. Membership functions for these flows are illustrated in Figures 14 through 16; they represent typical industrial variations. Tray efficiencies for the main column and side stripper were obtained using the methods recommended by Vital et al. (1982). The uncertainty in these quantities corresponds to the reported accuracy of the correlations. Membership functions for main column and side stripper efficiencies are given in Figures 17 and 18, respectively.

The stochastic uncertainties are listed in Table 14. Typical values were used for the heat transfer coefficients and the associated uncertainties. Uncertainties in the flooding velocities reflect the accuracy of the correlation as reported by van Winkle (1967).

This example problem contains four degrees of freedom, 13 constraints, and 14 uncertainties, so some simplification is in order. First, constraints were used to eliminate operating degrees of freedom. The product composition constraints and the cooling water exit temperature constraint are lower bounds on the utility consumption rates for all values of the side draw rate and the uncertain parameters. Because it is economically undesirable to overreflux a column in order to prevent the cool-

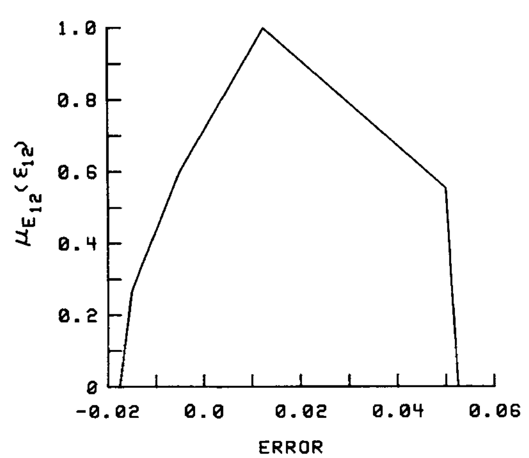

Figure 11. Membership function for error in benzene-toluene pair.

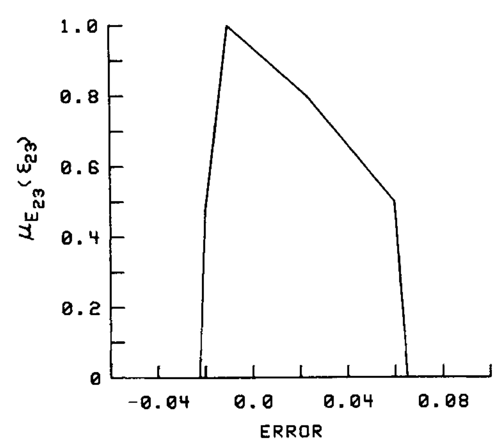

Figure 13. Membership function for error in toluene–*m*-xylene pair.

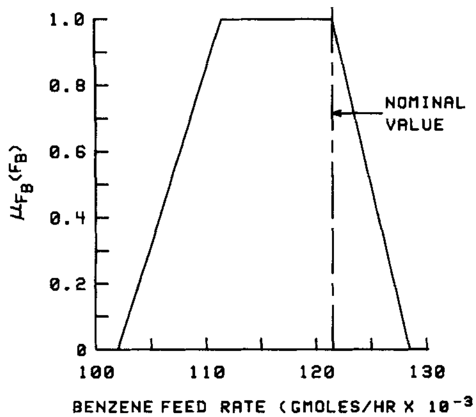

Figure 14. Membership function for benzene component flow in feed.

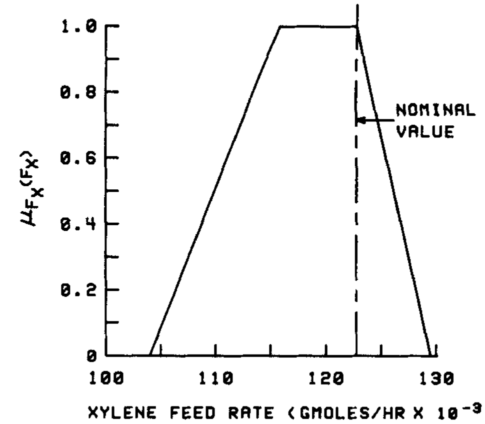

Figure 16. Membership function for *m*-xylene component flow in feed.

ing water from boiling, the product composition constraints were used to eliminate the cooling water flow, the main reboiler steam flow rate, and the side reboiler steam flow rate as degrees of freedom. If this simplification results in a significant possibility of failure due to excessive cooling water temperature, the assumption is wrong. But, the design is also inadequate and modifications are in order.

In order to identify the significant constraints and uncertainties, linear analysis was performed. Table 15 gives the relative magnitudes of the uncertainties based on the min-max formulation and the distance to the limiting value. This table indicates that none of the constraints is far enough from its limiting value to be ignored. Also, two of the uncertainties are insignificant: the side stripper tray efficiency and the side reboiler heat transfer coefficient.

Reliability and unreliability were calculated using a Monte Carlo sample size of 50. The results of the analysis are summarized in Table 16. The design has an excellent possibility of working, but the possibility of failure is also very great, which indicates that the design cannot compensate for all of the uncertainties. The dominant mode of failure is the limit on the steam pressure in the side reboiler. Because the overall unreliability equals the side reboiler failure possibility, fuzzy uncertainties appear to dominate the design. This observation is also sup-

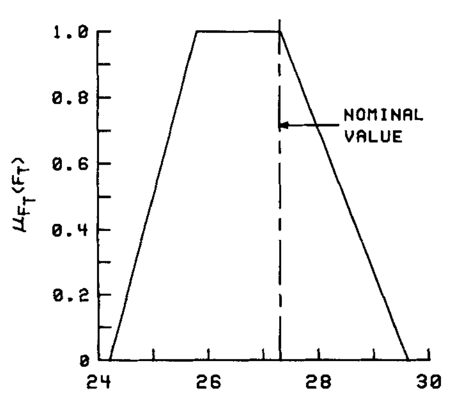

Figure 15. Membership function for toluene component flow in feed.

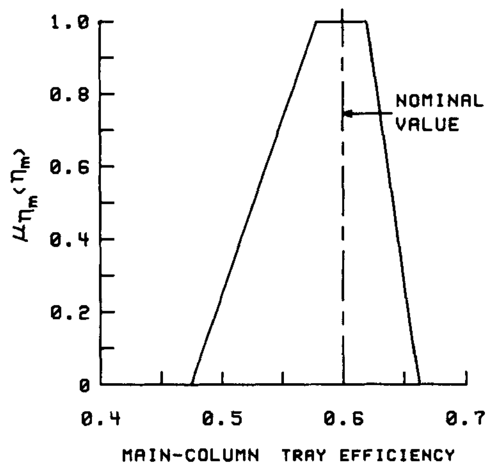

Figure 17. Membership function for main column tray efficiency.

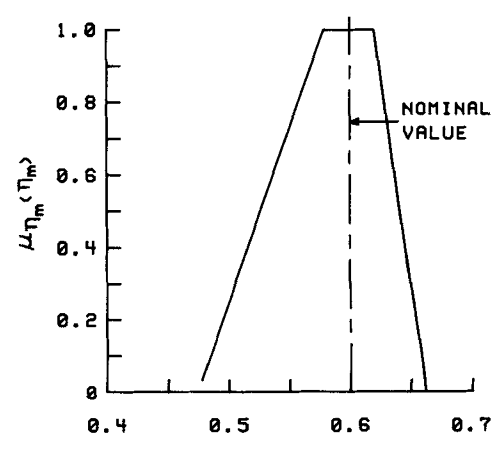

Figure 18. Membership function for side stripper tray efficiency.

Table 14. Stochastic Uncertainties for Benzene–Toluene–m-Xylene Separation Example

Uncertainty	Type of Distribution	Mean	Std. Dev.
Condenser U,* $J/m^2 \cdot h \cdot K$	Gamma	3,078,000	308,000
Main reboiler U, $J/m^2 \cdot h \cdot K$	Gamma	5,118,000	778,000
Side reboiler U, $J/m^2 \cdot h \cdot K$	Gamma	5,118,000	778,000
Main column flooding**	Normal	0	0.050
Side stripper flooding**	Normal	0	0.050
Cooling water temp., K	Normal	305.6	2.8

*U = heat transfer coefficient.
**Error = ln $(V_{f,correlation}/V_{f,actual})$.

ported by the linear analysis, which shows that the fuzzy uncertainties in the vapor-liquid equilibrium model have the greatest relative importance.

The failure possibility due to boiling of cooling water is very large, indicating that the assumption used to eliminate degrees of freedom is incorrect. In general, this erroneous assumption would result in an overestimation of the unreliability; but because the side reboiler steam pressure constraint is the dominant mode of failure, the overall value of unreliability is correct.

The large value of unreliability makes the proposed design unacceptable. A more reliable design was obtained by changing some of the equipment design parameters. With the exception of the cooling water exit temperature constraint, constraint violation possibilities can be reduced by increasing equipment sizes. Excessive cooling water temperature is the result of too much condenser area. One way of avoiding this constraint without increasing the failure possibilities of other constraints is to design a variable-area condenser. This change could be achieved by using several parallel exchangers to supply the required area. The design changes are summarized in Table 17.

The results of the reliability analysis for the alternate design

Table 16. Reliability and Unreliability for Benzene–Toluene–m-Xylene Separation Example

Reliability	1.000
Upper bound on unreliability	0.875
Mode Failure Possibilities	
Cooling water flow	0.190
Cooling water exit temp.	0.815
Reflux flow rate	0.598
Main column flooding	0.700
Main reboiler steam press.	0
Main reboiler steam flow	0.624
Side stripper flooding	0
Side reboiler steam press.	0.875
Side reboiler steam flow	0.473
Average Side draw rate, kmol/h	56.8

are given in Table 18. The unreliability was reduced to an acceptable level with these design changes. The constraints most likely to be violated in the alternate design are the main column flooding limit and the cooling water flow limit.

In this example, the major uncertainty is in the vapor-liquid equilibrium model, as indicated by the linear analysis, Table 15. The large safety factors required to accommodate this uncertainty can be reduced by improving the vapor-liquid equilibrium predictions, which in turn would require either additional experimental data or a better model. The linear analysis also indicates that the design is less sensitive to the benzene–m-xylene interaction than to the benzene-toluene or the toluene–m-xylene interactions. This observation suggests that experimental studies should concentrate on the benzene-toluene and the toluene–m-xylene pairs.

Summary and Conclusions

The concept of a fuzzy measure has been used to quantify design confidence in a way that considers the operability of the

Table 15. Linear Analysis of Benzene–Toluene–m-Xylene Separation Example

Uncertain Parameters	Constraints								
	Cooling Water Flow	Cooling Water Exit Temp.	Reflux	Main Column Flooding	Main Reboiler Steam Press.	Main Reboiler Steam Flow	Side Stripper Flooding	Side Reboiler Steam Press.	Side Reboiler Steam Flow
Benzene feed rate	0.214	0.215	0.085	0.211	0.075	0.225	0.029	0.078	0.031
Toluene feed rate	0.152	0.153	0.155	0.178	0.063	0.190	0.396	0.076	0.263
Xylene feed rate	0.026	0.026	0.026	0.051	0.014	0.041	0.775	0.075	0.117
VLE error for B/T*	0.664	0.665	0.665	0.690	0.242	0.735	0.578	0.651	0.566
VLE error for B/X	0.066	0.066	0.066	0.274	0.097	0.294	0.066	0.088	0.090
VLE error for T/X	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Main column tray eff.	0.105	0.105	0.105	0.128	0.036	0.108	0.008	0.013	0.013
Side strip. tray eff.	0	0	0	0	0	0	0	0	0
Condenser U**	0.357	0.792	0	0	0	0	0	0	0
Main reboiler U	0	0	0	0	0.357	0.010	0	0	0
Side reboiler U	0	0	0	0	0	0	0	0.077	0.031
Main column flooding	0	0	0	0.304	0	0	0	0	0
Side strip. flooding	0	0	0	0	0	0	0.518	0	0
Cooling water temp.	0.284	0.174	0	0	0	0	0	0	0
Std. dist. to limit	0.357	0.134	0.238	0.206	0.414	0.238	0.418	0.100	0.277

*B, benzene; T, toluene; X, m-xylene
**U = heat transfer coefficient.

Table 17. Modifications to Benzene–Toluene–*m*-Xylene Separation Design

Design Parameter	Base Design	Alternate Design
Main Column		
Column dia., m	1.65	1.77
Reflux capacity, kmol/h	318	350
Condenser		
Area, m^2	83.6	83.6
Configuration	Single unit	Multiple unit
Main Reboiler		
Area, m^2	325	307
Max. steam flow, kg/h	6,360	7,270
Side Reboiler		
Area, m^2	11.6	23.2
Max. steam flow, kg/h	841	886

process and the nature of the uncertainties. Two measures of design confidence, called design reliability and unreliability, were defined; they correspond to the chances of the design succeeding or failing, respectively. The analysis was developed for designs subject to random uncertainties, designs subject to fuzzy uncertainties, and designs subject to both random and fuzzy uncertainties.

If a design is subject only to stochastic uncertainties, design reliability is based on a probability measure, and it is equal to the probability that the design will work. Because rigorous calculation of design reliability is difficult, a procedure was developed for estimating its value. The probabilistic analysis of design reliability can be combined easily with statistical decision theory to study the economics of overdesign.

Fuzzy set theory is a different and useful method of representing uncertainties. By manipulating the fuzzy sets of parameter uncertainties and by applying possibility theory, a possibilistic measure of design confidence has been developed which has properties that are quite different from a probabilistic measure. Fuzzy design reliability and unreliability correspond to the probabilistic analysis in that they measure the chance of success and failure; but, in addition, they can be used to measure the sufficiency of information and the adequacy of model predictions.

The ignorance factor measures the sufficiency of information or the need for data. In the case of model error, ignorance measures the availability of pertinent data for testing a model. A high ignorance factor indicates a need to refine information by experimentation or other methods. Having large values of fuzzy reliability and unreliability for a design indicates that a model used in the analysis is not precise enough to predict whether the design is adequate. This condition can also suggest experimental studies needed to improve the quality of information or to reduce uncertainty by eliminating the need to rely on imprecise models.

Table 18. Reliability and Unreliability for Alternate Benzene–Toluene–*m*-Xylene Separation Design

Reliability	1.000
Upper bound on unreliability	0.319
Mode Failure Possibilities	
Cooling water flow	0.168
Cooling water exit temp.	0
Reflux flow rate	0.076
Main column flooding	0.205
Main reboiler steam press.	0
Main reboiler steam flow	0
Side stripper flooding	0.005
Side reboiler steam press.	0
Side reboiler steam flow	0.045
Avg. side draw rate, kmol/h	56.8

Design criteria based on fuzzy reliability and unreliability can mimic the decision process of an engineer. They can be used to distinguish between feasible designs, those with high reliability and low reliability; infeasible designs, those for which the opposite is true; and promising designs. A promising design is one that is found to be economically desirable when the analysis is based on the best estimate of performance, but which has a large ignorance factor, or a reliability and unreliability that are nearly equal. A promising design requires further study.

The analysis of designs subject to both stochastic and fuzzy uncertainties results in a general measure of design confidence in a way that retains the features of probability theory and fuzzy set theory. The general measure allows the engineer to combine quantitative measures of uncertainties, in the form of probability distributions, with qualitative knowledge of uncertainties, in the form of fuzzy sets, to access the adequacy of a given design.

The values of a systematic method of measuring design confidence are:

- To determine the ability of a design to compensate for uncertainty
- To indicate how a design can be modified to compensate for uncertainty
- To suggest experiments to reduce the uncertainty and, hence, the required overdesign

Computation of design reliability can also yield information on the expected uncertainties in utility consumption, which in turn can be used to design utility plants.

Notation

d_i = distance from constraint i
$\boldsymbol{d}$ = vector of equipment design parameters
E_i = set of even number of constraint failures
f_l = liquid fraction of distillation feed
F = flexibility index
$\mathcal{F}$ = defined by Eq. 18
F_i = set of operating conditions violating constraint i
F_T = set of operating conditions violating any constraint
$g_i(\cdot)$ = constraint i
$h(\cdot)$ = general fuzzy measure
I = ignorance factor
m_f = distillation feed rate
M = set of parameters satisfying all constraints
N = set of parameters violating at least one constraint
N_i = set of parameters violating constraint i
O_i = set of odd number of constraint failures
$p(\cdot)$ = probability density function
$\boldsymbol{p}$ = vector of uncertain parameters
$\boldsymbol{p}_f$ = vector of fuzzy parameters
$\boldsymbol{p}_s$ = vector of stochastic parameters
P = set of all possible parameters
P_s = universe of stochastic parameters
$r_{i,j}$ = relative contribution of parameter j to constraint i variance
R = design reliability or gas constant
$\mathcal{R}$ = fuzzy design reliability
$\mathcal{R}^*(\boldsymbol{u})$ = fuzzy design reliability for operating conditions $\boldsymbol{u}$

R = fuzzy set of feasible stochastic parameters
s_j = scaling factor for constraint j
T = temperature
$T(\delta)$ = set of parameters, Eq. 9
$\bar{T}(\delta)$ = complement of $T(\delta)$
u = vector of operating degrees of freedom
$u^*(p)$ = most reliable operating conditions
U = design unreliability, or heat transfer coefficient
$\mathcal{U}$ = fuzzy design unreliability
$\mathcal{U}^*(u)$ = fuzzy design unreliability for operating conditions u
$\mathcal{U}_j^*(u)$ = possibility of constraint j failure
U = fuzzy set of infeasible stochastic parameters
v_k = molar volume of component k
x_k = mole fraction of component k

Greek letters

$\alpha(\cdot)$ = defined by Eq. 4
γ_k = activity coefficient of component k
δ = standard distance to constraint limit
δ_k = solubility parameter for component k
ϵ = error in regular solution theory
$\mu_A(\cdot)$ = membership function for set A
$\pi(\cdot)$ = possibility distribution
σ = standard deviation
Φ_k = volume fraction of component k

Subscripts

g_i = constraint g_i
i, j = vector element indices
p_j = parameter p_j

Appendix: Upper and Lower Bounds on Design Reliability

Let E_i be sets of failure due to an even number of constraint violations, and O_i be sets of failure due to an odd number of constraint violations. By definition, the E_i are disjoint from the O_i, so reliability can be written as follows:

$$R = 1 - Pr\{\cup E_i\} - Pr\{\cup O_i\} \tag{A1}$$

Probabilities are nonnegative, so

$$R \leq 1 - Pr\{\cup O_i\} \tag{A2}$$

$\cup O_i$ is equal to the set of parameters for which $\mathcal{F}(d, p)$ is less than zero, which implies that

$$Pr\{\cup O_i\} = Pr\{\mathcal{F}(d, p) < 0\} \tag{A3}$$

Combining Eqs. A2 and A3 establishes an upper bound on reliability:

$$R \leq 1 - Pr\{\mathcal{F}(d, p) < 0\} = Pr\{\mathcal{F}(d, p) \geq 0\} \tag{A4}$$

To evaluate a lower bound on reliability, the probability of failure is written as follows:

$$Pr\{N\} = \sum_{i=1}^{n} Pr\{N_i\} - \sum_{k=2}^{n} (k-1) Pr\{\text{exactly } k\} \tag{A5}$$

$Pr\{\text{exactly } k\}$ is the probability that exactly k constraints are violated simultaneously. The probability of an even number of constraint violations can be determined from Eqs. A1, A2, and A4.

$$Pr\{\cup E_i\} = Pr\{\mathcal{F}(d, p) \geq 0\} - R \tag{A6}$$

An alternate way of writing this probability is

$$Pr\{\cup E_i\} = \sum_{k=1}^{[n/2]} Pr\{\text{exactly } 2k\} \tag{A7}$$

Combining Eqs. A5, A6, and A7 yields the following expression for failure probability:

$$\begin{aligned} Pr\{N\} = \sum_{i=1}^{n} Pr\{N_i\} - Pr\{\mathcal{F}(d, p) \geq 0\} + R \\ - \sum_{k=1}^{[(n-1)/2]} (k-1) Pr\{\text{exactly } 2k+1\} \\ - \sum_{k=1}^{[n/2]} k Pr\{\text{exactly } 2k\} \end{aligned} \tag{A8}$$

Because probabilities are nonnegative, Eq. A8 implies the following inequality.

$$Pr\{N\} \leq \sum_{i=1}^{n} Pr\{N_i\} - Pr\{\mathcal{F}(d, p) \geq 0\} + R \tag{A9}$$

This inequality leads to a lower bound on reliability.

$$R \geq \left[1 + Pr\{\mathcal{F}(d, p) \geq 0\} - \sum_{i=1}^{n} Pr\{N_i\}\right] \Big/ 2 \tag{A10}$$

Literature Cited

Alatiqi, I. M., "Composition Control of Distillation Systems Separating Ternary Mixtures with Small Intermediate Concentrations," Ph.D. Diss., Lehigh Univ. (1985).

Aspen Technology, Inc., *Aspen User's Guide*, Cambridge, MA (1981).

Bolles, W. L., and J. R. Fair, "Performance and Design of Packed Columns," *3rd Int. Symp. on Distillation* (1979).

Chen, M. S. K., L. E. Erickson, and L. T. Fan, "Consideration of Sensitivity and Parameter Uncertainty in Optimal Process Design," *Ind. Eng. Chem. Process Des. Dev.*, **9**, 514 (1970).

Cox, N. D., "Tolerance Analysis by Computer," *J. Quality Technol.*, **11**, 80 (1979).

Dubois, D., and H. Prade, *Fuzzy Sets and Systems: Theory and Applications*, Academic Press, New York (1980).

Eckert, J. S., "Selecting the Proper Distillation Packing," *Chem. Eng. Prog.*, **66**, 39 (Mar., 1970).

Freeman, R. A., and J. L. Gaddy, "Quantitative Overdesign of Chemical Processes," *AIChE J.*, **21**, 436 (1975).

Glitsch, Inc., *Ballast Tray Design Manual*, Glitsch, Inc., Dallas (1984).

Kubic, W. L., "The Effect of Uncertainties in Physical Properties on Chemical Process Design," Ph.D. Diss., Lehigh Univ. (1986).

Kubic, W. L., and F. P. Stein, "The Application of Fuzzy Set Theory to Uncertainty in Physical Property Models," *Fluid Ph. Equil.*, **30**, 111 (1986).

Lashmet, P. K., and S. Z. Szczepanski, "Efficiency Uncertainty and Distillation Overdesign Factors," *Ind. Eng. Chem. Process Des. Dev.*, **13**, 103 (1974).
Macchietto, S., G. Maduabeuke, and R. Szczepanski, "Exact Determination of Process Sensitivity to Physical Properties," *Fluid Ph. Equil.*, **29**, 59 (1986).
Park, S. W., and D. M. Himmelblau, "Error in the Error Propagation Formula," *AIChE J.*, **26**, 168 (1980).
Reid, R. C., J. M. Prausnitz, and T. K. Sherwood, *The Properties of Gases and Liquids*, McGraw-Hill, New York (1977).
Swaney, R. E., and I. E. Grossmann, "An Index for Operational Flexibility in Chemical Process Design. 1: Formulation and Theory," *AIChE J.*, **31**, 621 (1985).
Sullivan, S. T., and V. W. Uhl, "Uncertainty Analysis in the Appraisal of Capital Investment Projects," *Uncertainty Analysis for Engineers, AIChE Symp. Ser.*, **220**(78), 10 (1982).
van Winkle, M., *Distillation*, McGraw-Hill, New York (1967).
Vital, T. J., S. Grossel, and P. Olsen, "Distillation and Absorption Tray and Packed Tower Efficiencies: A Critical Review of the Literature and Recommended Predictive Methods," *AIChE Summer Nat. Meet.*, Cleveland, OH (1982).
Zadeh, L. A., "Fuzzy Sets," *Information and Control*, **8**, 338 (1965).
———, "The Concept of a Linguistic Variable and Its Application to Approximate Reasoning. 1," *Inf. Sci.*, **8**, 199 (1975).
———, "Fuzzy Sets as a Basis for a Theory of Possibility," *Fuzzy Sets, Systems*, **1**, 3 (1978).

Manuscript received in three parts Mar. 12, Apr. 6 and May 5 of 1987, and revision received Nov. 19, 1987.